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On the semiclassical propagator for the anharmonic oscillator

M Fusco Girard

Dipartimento di Fisica Teorica and Istituto Nazionale di Fisica della Materia, Università di Salerno, 84100 Salerno, Italy

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Abstract. The semiclassical propagator for the one-dimensional anharmonic oscillator is investigated, in the configuration space, by means of the so-called Van Vleck formula, which expresses it as a sum over all the denumerably infinite classical paths connecting given points in the same time. Analytical formulae for the paths' contributions are given, together with some numerical results. It is shown that in the general case the amplitudes of the contributions asymptotically approach the same value, while the phases oscillate; the Van Vleck series therefore does not converge, but in the generic case it can be resummed. Finally, the conditions under which one of the paths gives a dominant contribution to the semiclassical propagator are discussed.

1. Introduction

As is well known, in the semiclassical approximation the Feynman's path integral [1] for the propagator reduces to a sum [2, 3], to which all the classical paths, going from the initial point x_A to the final point x_B in the same time T, contribute.

In the generic case, there are many classical trajectories, with different energies, connecting the given points in time T, so that it is interesting to investigate how the different paths take part in the quantum amplitude in this approximation, their contributions being in general different: in fact, the weight of the contribution is related to the number of neighbouring paths which constructively interfere, and this number is *a priori* different for the various extremals of the action.

The aim of this paper is to present the results of such an investigation for the one-dimensional anharmonic oscillator, for which a quartic term is added to the harmonic potential in the Lagrangian. Due to its intrinsic interest and being one of the simplest nonlinear systems, the anharmonic oscillator was and still is an active field of research, in particular for what pertains to the relationship between classical and quantum mechanics [4-8]. For our purposes, it is the simplest system with denumerably infinite classical paths connecting given points x_A and x_B in time T, while for the corresponding unperturbed system, the harmonic oscillator, there is in general only one such path.

Various aspects of the problem which we are interested in are discussed in [2] and [9], where the validity of the usual perturbative expansion is particularly investigated. In the present paper we give the explicit analytical expressions for the classical paths' contributions to the semiclassical propagator as a function of the coordinates and time; the actual calculation needs the numerical determination of the path's energy, and an example of this computation is presented in the last part of the paper. Moreover, from the analytical results it follows that in general the various classical trajectories, which may be labelled by some integer n, contribute with amplitudes which asymptotically approach the same value, with increasing n; in this way the series for the semiclassical propagator does not converge, but it is shown that in general it can be Cesaro resummed. Finally, *it is shown that two countable sequences of times correspond to any choice* of x_A and x_B, in such a way that the amplitude of one of the related paths diverges, so that the relative contribution to the total propagator has predominant importance with respect to the others.

The paper is organized in the following manner: in section 2 the analytical results are given, while section 3 is devoted to the presentation of the numerical results, together with the investigation of the asymptotic behaviour and the divergences in the amplitude.

2. Analytical results

According to the so-called Van Vleck formula [3, 10], the semiclassical approximation K_{WKB} for the propagator $K(x_B, t_B; x_A, t_A)$ is given, in the one-dimensional case, by

$$K_{\mathrm{WKB}} = \sum_\alpha \left(\frac{\mathrm{i}}{2\pi\hbar}\right)^{1/2} \left|\frac{\partial^2 S_\alpha}{\partial x_A\, \partial x_B}\right|^{1/2} \mathrm{e}^{\mathrm{i}(S_\alpha/\hbar - n_\alpha \pi/2)} \tag{1}$$

here S_α is the action for the αth classical path from the initial point x_A at time t_A to the final point x_B at time t_B, i.e.

$$S_\alpha = \int_{t_A}^{t_B} L(x_\alpha(t), \dot{x}_\alpha(t), t)\, \mathrm{d}t \tag{2}$$

and n_α is the number of focal points along the path (see below for details). The one-dimensional anharmonic quartic oscillator is described by the Lagrangian

$$L = \tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}\omega^2 x^2 - \tfrac{1}{4}\lambda x^4 \qquad (\lambda > 0). \tag{3}$$

The solutions of the equation of motion are expressed in terms of Jacobi elliptic functions (see, for instance, [9]).

In order to compute K_{WKB} for the Lagrangian (3) we have to find all the classical paths which connect the initial and the final points in the time $T = t_B - t_A$. These are determined by specifying their energy E and the directions of the initial and final momenta p [11]. In general the paths may be divided into four classes according to the signs of the momenta: the first class includes the paths with $p_A > 0$ and $p_B > 0$; for the second class $p_A > 0$ and $p_B < 0$, for the third class $p_A < 0$ and $p_B > 0$, and finally for the fourth class $p_A < 0$ and $p_B < 0$. For each class, the paths may be labelled by the number n of complete oscillations, and their energies $E(x_A, x_B, T)$ are the solutions of the equation

$$T = \pm \int_{x_A}^{x_B} \frac{\mathrm{d}x}{[2E - \omega^2 x^2 - \frac{1}{2}\lambda x^4]^{1/2}} \tag{4}$$

where, for each part of a path, the sign has to be chosen according to the sign of $\mathrm{d}x$, so that the time t is non-decreasing along the path.

Let us define

$$D(x) \equiv [2E - \omega^2 x^2 - \tfrac{1}{2}\lambda x^4]^{1/2} \tag{5}$$

and

$$t^K = t^K(E) \equiv \int_0^{x_K} \frac{\mathrm{d}x}{D(x)} \tag{6}$$

where $K = A, B$ and the integration is done on the direct path from the origin 0 to the point x_K (t^K should not be confused with t_K). t^K is expressed in terms of the elliptic integral of first kind $F(\varphi, k)$ [12]

$$F(\varphi, k) = \int_0^{\varphi} \frac{\mathrm{d}\alpha}{[1 - k^2 \sin^2 \alpha]^{1/2}} \tag{7}$$

by

$$t^K = \frac{1}{\Lambda R^{1/2}} F(\gamma_K, r) \tag{8}$$

where

$$\Lambda = (\tfrac{1}{2}\lambda)^{1/2} \tag{9}$$

$$R = a^2 + b^2 \tag{10}$$

$$a^2 = \omega^2/\lambda + [\omega^4/\lambda^2 + 4E/\lambda]^{1/2} \tag{11}$$

$$b^2 = -\omega^2/\lambda + [\omega^4/\lambda^2 + 4E/\lambda]^{1/2} \tag{12}$$

$$\gamma_K = \sin^{-1} \frac{x_K}{b} \left[\frac{R}{a^2 + x_K^2}\right]^{1/2} \tag{13}$$

$$r = b/R^{1/2}. \tag{14}$$

b and $-b$ are the coordinates of the turning points. t^K as a function of the energy E starts from a finite value at $E = E_{\min}$, which is the minimum value of the energy such that the particle may reach the point x_K, and monotonically decreases to zero when E goes to infinity. The period $P(E)$ of the motion, which is given by

$$P(E) = 4 \int_0^b \frac{\mathrm{d}x}{D(x)} = (4/\Lambda) R^{-1/2} \mathrm{K}(r) \tag{15}$$

where $\mathrm{K}(r) = F(\pi/2, r)$ denotes the complete elliptic integral of the first kind, starts from $2\pi/\omega$ at $E = 0$ and is a monotonic decreasing function, going to zero when E tends to infinity.

For the first class of paths, with the definitions above, (4) can be written as

$$T = t^B(E) - t^A(E) + nP(E). \tag{16}$$

Given x_A and x_B, the RHS of (16) is a function of E, starting from a finite value (which increases with n) at $E = E_{\min}$, and monotonically decreasing to zero; this implies that the equation (16) has a unique solution $E_n(x_A, x_B, T)$ for each integer n greater than a minimum $n_{\min}$. The energy values E_n make an increasing unbounded sequence. Analogously, for the second class, (4) becomes

$$T = (\tfrac{1}{2}P - t^B) - t^A + nP = (n + \tfrac{1}{2})P - t^B - t^A \tag{17}$$

and for the third and fourth class we have, respectively

$$T = (n + \tfrac{1}{2})P + t^B + t^A \tag{18}$$

$$T = (n + 1)P - t^B + t^A. \tag{19}$$

For a given energy E, the corresponding action S is computed from the relation

$$S(x_A, x_B, T) = W(x_A, x_B, E) - ET \tag{20}$$

which holds true for every conservative system, where W is given by

$$W(x_A, x_B, E) = \pm \int_{x_A}^{x_B} p \, dx. \tag{21}$$

For the anharmonic oscillator

$$W = \pm \int_{x_A}^{x_B} D(x) \, dx \tag{22}$$

where the sign along the various parts of the trajectory has to be chosen in order to obtain W as a non-decreasing function along the path. Let us define

$$w^K \equiv \int_0^{x_K} D(x) \, dx \tag{23}$$

where $K = A, B$ and the integration is done along the direct path of energy E from the origin to x_K; this gives the result [12]

$$w^K = \Lambda \int_0^{x_K} [(a^2 + x^2)(b^2 - x^2)]^{1/2} \, dx = (\Lambda/3) R^{1/2} \left\{ a^2 F(\gamma_K, r) - (2\omega^2/\lambda) E(\gamma_K, r) + (x_K/3)(x_K^2 + 2a^2 - b^2) \left[\frac{b^2 - x_K^2}{a^2 + x_K^2} \right]^{1/2} \right\} \tag{24}$$

where $E(\gamma_K, r)$ denotes the elliptic integral of second kind

$$E(\varphi, k) = \int_0^{\varphi} [1 - k^2 \sin^2 \alpha]^{1/2} \, d\alpha. \tag{25}$$

For the first class one obtains

$$W = w^B - w^A + nJ \tag{26}$$

where nJ is the contribution to W by the n complete oscillations, and J is the action variable

$$J(E) = \oint p \, dx = (4\Lambda/3) R^{1/2} [a^2 \mathrm{K}(r) - (2\omega^2/\lambda) \mathrm{E}(r)]. \tag{27}$$

$\mathrm{E}(r)$ denotes the complete elliptic integral of second kind. In this way, the action for the first class of paths is given by

$$S = S(x_A, x_B, T) = nJ + w^B - w^A - ET. \tag{28}$$

Analogous expressions for the other classes of paths can be easily obtained.

In order to compute the amplitude of the path's contribution to the semiclassical propagator, as given by (1), we need the second derivative of the action with respect to the initial and final coordinates, i.e.

$$S_{x_A x_B} = \frac{\partial^2 S}{\partial x_A \, \partial x_B} \tag{29}$$

(derivatives will be denoted by means of suffixes). From (28) one has for the first class of paths

$$S_{x_A} = (nJ_E + w_E^B - w_E^A - T)E_{x_A} - w_{x_A}^A \tag{30}$$

and, since $J_E = P$, $w_E^A = t^A$ and $w_E^B = t^B$, the sum in parentheses in the last equation is identically zero with respect to x_A, x_B and T, due to equation (16). In this way

$$S_{x_A,x_B} = w_{x_A,E}^A E_{x_B}. \tag{31}$$

The same result holds for the second class, while for the third and fourth classes the sign in the RHS of (31) is positive. From equation (23) it follows that

$$w_{x_A,E}^A = D(x_A)^{-1} = (1/\Lambda)[(a^2+x_A^2)(b^2-x_A^2)]^{-1/2} \tag{32}$$

and by deriving equation (16) with respect to x_B, for the first class one has

$$E_{x_B} = -t_{x_B}^B[nP_E + t_E^B - t_E^A]^{-1} \tag{33}$$

where, as follow from equations (6) and (14) [12]

$$t_{x_B}^B = D(x_B)^{-1} \tag{34}$$

$$t_E^K = -\int_0^{x_K} \frac{\mathrm{d}x}{D(x)^3} = -(1/\Lambda)^3\left\{\frac{1}{a^2b^2R^{3/2}}[a^2F(\gamma_K, r) - (2\omega_K^2/\lambda)E(\gamma_K, r)] + \frac{x_K}{b^2R[(a^2+x_K^2)(b^2-x_K^2)]^{1/2}}\right\} \tag{35}$$

with $K = A, B$, while P_E is easily obtained from (15) as

$$P_E = \frac{4}{\Lambda^{3/2}b^2R^{3/2}}\left[-\mathsf{K}(r) + \frac{(2\omega^2/\lambda)}{a^2}\mathsf{E}(r)\right]. \tag{36}$$

As for the other classes, one has only to change in (33) n into $(n+\frac{1}{2})$ for the second and third classes and into $(n+1)$ for the fourth class, and to choose the correct signs for t_E^B and t_E^A.

Finally, in order to compute the paths' contributions to the semiclassical propagator, the number n_α of focal points is needed. For a complete discussion of this point see [3, 13, 14]. We will here only recall that a point x_C, t_C along the path $x(t)$ leaving x_A, t_A is called focal or conjugate to the initial point if there it results in $\delta x = 0$, where $\delta x[x(t)]$ denotes the solution of the Jacobi equation, i.e. the infinitesimal deviation from the path, with $\delta x(x_A, t_A) = 0$ but with time derivative different from zero in the initial point. The deviation δx could be obtained as a function of time, by differentiating the solution of the motion's equation with respect to the integration's constants. However, in order to count the zeros of δx, it is simpler to use the following approach. Let us consider the first class of paths; their equations $x = x(t)$ are obtained from

$$t - t_A = \pm\int_{x_A}^{x} \frac{\mathrm{d}x}{D(x)} = t^x - t^A + nP \tag{37}$$

where $t - t_A$ is the total time from x_A to the point x, and t^x is the time to go from the origin to x along the direct path. By differentiating (37) with respect to x and E we have

$$\delta x = -\frac{(t_E^x - t_E^A + nP_E)}{t_x^x}\,\delta E. \tag{38}$$

By putting here $E = E_i$, one of solutions of (16), and after choosing a value for δE, we obtain the corresponding deviation $\delta x(x)$ from the path of energy E_i leaving x_A at time t_A. The deviation is parametrized by the corresponding coordinate of the unvaried path (i.e. x and δx in (38) refer to the same time t). The quantities in (38) are given by equations (34) to (36), with x_K replaced by x.

By comparing the last equation with equations (31) and (33) one recovers the well known result [14] that the Jacobi field δx is proportional to the inverse of S_{x_A,x_B}, so that in a focal point the second derivative of the action S diverges, together with the corresponding path's contribution to the propagator. The conjugate points are given by the zeros of the RHS of (38) and their study is simple, although their exact determination can be done, in general, only numerically. At the turning points b and $-b$ the denominator $t^x_x = D(x)^{-1}$ in (38) diverges, but this does not correspond to a vanishing δx, because the divergence is compensated by an analogous divergence of the third term in t^x_E, as given by (35) with the substitution $x_K \to x$. So, the zeros of δx reduce to the zeros of the numerator.

Let us suppose for simplicity $x_A = 0$, $t_A = 0$ and $x_B > 0$, and let us follow the numerator in (38) while the current point x, moving from the origin, oscillates between b and $-b$. For $0 < x < b$, the numerator reduces to $-t^x_E$, which is zero at $x = 0$ and increases monotonically to ∞ for $x \to b$, as seen from the first equality in (35). When x decreases from b to 0, the numerator becomes $t^x_E - (1/2)P_E$, which is zero in some point x_C, because $P_E < 0$. For x going from 0 to $-b$, the numerator is $-(1/2)P_E - t^x_E$, which is always greater than zero, while it will pass through zero for x increasing from $-b$ to 0, being there $t^x_E - P_E$. This analysis immediately can be extended to the other oscillations of the current point, and it shows that there it will be a focal point x_C along each arc of the path corresponding to x decreasing from b to 0 or increasing from $-b$ to 0, while there are no focal points for the other arcs. So, every complete oscillation of the path introduces two more focal points. From the above analysis it follows that for the first and third classes the number of focal points is equal to $2n$ or $2n+1$, respectively; for the other classes, the last arc of the path, corresponding to x decreasing from b to x_B, will or will not contain a further focal point, according to $x_B < x_C$ or $x_B > x_C$; this can be only numerically established. Finally, it is easy to show that, as the energy increases, the focal points approach the turning points $+b(E)$ and $-b(E)$.

3. Numerical results, asymptotic behaviour and discussion

The results of the previous section have been used for the numerical computation of the paths' contributions to the semiclassical propagator, according to (1). An example of such a computation is given in table 1, where the amplitude and phase of the contributions are reported for the first few values of n, together with the corresponding energy, with the following choice of parameters: $x_A = 0$, $x_B = 1$, $T = 1$, $\omega = 1$, $\lambda = 0.1$. The energies were obtained by numerically solving equations (16) to (19) and these values were then used to compute the amplitude and the phase of the contributions, according to the procedure detailed in the previous section. The phases are reduced to the interval $[-\pi, \pi]$, and we used the value $\hbar = 1$ for Planck's constant.

For the corresponding unperturbed harmonic oscillator, with the same values of parameters x_A, x_B, T and ω, the (unique) path's energy is $E = 0.706\,14$ and the propagator's amplitude is $A = 0.434\,90$. These values only slightly differ from the

Table 1. The energy of the path and the amplitude and the phase of the corresponding contribution to the semiclassical propagator are given, for each class, for the first few values of the number n of complete oscillations. The parameter values are $x_A = 0$, $x_B = 1$, $T = 1$, $\omega = 1$ and $\lambda = 0.1$.

n	Class	Energy	Amplitude	Phase
0	1	0.715 42	0.439 19	−2.301 27
0	2	379.025 04	0.588 01	1.630 51
0	3	502.339 67	0.565 02	1.938 35
0	4	7 190.709 25	0.569 09	−1.638 77
1	1	7 682.731 33	0.564 28	−2.387 23
1	2	37 450.507 7	0.566 30	2.730 05
1	3	38 557.423 56	0.564 22	2.356 10
1	4	119 519.466 29	0.565 36	−0.389 18
2	1	121 487.277 35	0.564 20	1.024 01
2	2	293 102.869 94	0.564 93	−2.011 28
2	3	296 177.558 03	0.564 19	2.599 02
2	4	609 250.356 31	0.564 70	−0.457 28
3	1	613 677.898 57	0.564 19	2.476 23
3	2	1 130 355.924 16	0.564 56	−1.354 53
3	3	1 136 382.296 10	0.564 19	1.311 23
3	4	19 301 157.934 77	0.564 48	1.212 84

corresponding ones for the lowest energy path of the anharmonic oscillator, as seen in the first row of table 1.

As clearly seen from the table, after some initial fluctuation, the amplitude of the contributions very quickly stabilizes to a constant value, while the phases irregularly oscillate. Moreover, the energy values quickly increase with the number n of complete oscillations. This behaviour is also found for a generic choice of the initial and final points and of the time T, as easily follows from the analysis of the asymptotic behaviour of the various quantities involved. Indeed, for $E \gg 1$, from equations (8) to (15) one has

$$a^2 \sim b^2 \sim (2E)^{1/2}/\Lambda \tag{39}$$

$$r \sim 1/\sqrt{2} \tag{40}$$

$$t^B \sim (2E)^{-1/2} \tag{41}$$

$$P \sim CE^{-1/4} \tag{42}$$

where C is given by

$$C = (2)^{5/4}(\Lambda)^{-1/2}\mathrm{K}(1/\sqrt{2}) \tag{43}$$

so that, as seen from equations (16) to (19), the energy E grows with n as

$$E \sim (Cn/T)^4. \tag{44}$$

Moreover, from equations (32) to (36), one has

$$w^A_{x_A,E} \sim t^B_{x_B} \sim (1/\sqrt{2})E^{-1/2} \tag{45}$$

$$t^B_E \sim -(1/\sqrt{2})E^{-3/2} \tag{46}$$

$$P_E \sim -(C/4)E^{-5/4} \tag{47}$$

so that, for $n \gg 1$, the amplitude A_n of the path's contribution to the propagator is

$$A_n \sim (2\pi\hbar)^{-1/2}E^{-1}/(2nP_E) \sim (2\pi\hbar)^{-1/2}(2/T)^{1/2}. \tag{48}$$

Thus, in the general case, all the classical trajectories connecting the given points in the same time, give, in modulus, an identical asymptotic contribution to the semi-classical propagator. This is not trivial because, according to the Feynman's approach, all the paths, including the non-classical ones, equally contribute in modulus to the exact propagator; on the other hand the amplitude of the contribution of a classical trajectory to the semiclassical propagator is a measure of the number of neighbouring constructively interfering paths, and this number is *a priori* different for the various classical trajectories.

According to the Feynman's approach [1], the various classical trajectories present interfering alternatives to a semiclassical particle travelling between the given points in time T and the modulus squared of the amplitude gives the probability of the alternative; in the generic case all these classical paths therefore are (asymptotically for $n \gg 1$, and in practice very soon) equiprobable.

The last equation (48) also implies that the series for the semiclassical propagator, as given by (1), does not converge; however, as will be shown, in the general case the series can be resummed. According to equation (48), for $n \gg 1$, the contribution of the αth classical path to the semiclassical propagator K_{WKB} is given by

$$(2\pi\hbar)^{-1/2}(2/T)^{1/2}\,e^{i(S_\alpha/\hbar - n_\alpha \pi/2)} \tag{49}$$

where, according to (28), S_α can be written in the form

$$S_\alpha = (n+\delta_i)J_\alpha + \sigma_i w^B - w^A - E_\alpha T \tag{50}$$

with δ_i equal to 0 for the first class of paths, $\frac{1}{2}$ for the second and the third classes, and 1 for the fourth class, and σ_i is 1 for the first and the third classes, and -1 for the second and the fourth classes. In order to investigate the asymptotic behaviour of the exponents in (49) we have to evaluate the quantities occurring in (50) as powers of n, up to the order n^0. For this purpose, equations (16) to (19) for the energies have to be solved by searching for solutions in the form

$$E^{1/4} = \alpha n + \beta + \gamma/n. \tag{51}$$

We need also to improve the approximation for $P(E)$ as given by (42), and a straightforward calculation shows that

$$P(E) = CE^{-1/4} - \frac{\omega^2 C}{8\Lambda}E^{-3/4} + O(E^{-5/4}). \tag{52}$$

By inserting (41), (51) and (52) into equations (16) to (19) and by equating coefficients of powers of n one easily gets

$$E^{1/4} = (n+\delta_i)\frac{C}{T} + \frac{1}{n}\left(\frac{-\omega^2}{8\Lambda} + \frac{\sigma_i t^B - t^A}{C}\right) \tag{53}$$

therefore, for $n \gg 1$, and by including the zero-order term, the energy E of the path is given by a fourth-order polynomial in n

$$E = E_4 n^4 + E_3 n^3 + E_2 n^2 + E_1 n + E_0. \tag{54}$$

Moreover, the expansion of $J(E)$, as given by (27), for large energies is

$$J(E) = \tfrac{4}{3}CE^{3/4} + DE^{1/4} + O(E^{-3/4}) \tag{55}$$

with D = constant, and finally, from our discussion about the focal points, it follows that for $n \gg 1$ one can assume that the number n_α is given by

$$n_\alpha = 2(n+\delta_i). \tag{56}$$

By using equations (50) to (53) into (49) it follows that, for large values of n, the classical paths with n complete oscillations give the following contributions to the semiclassical propagator

$$(2\pi\hbar)^{-1/2}(2/T)^{1/2}\,e^{i/\hbar(\sigma_i w^B - w^A)+2\pi i\Phi(n)} \tag{57}$$

where $\Phi(n)$ is a fourth-order polynomial in n, with coefficients depending on T, t^A, t^B, δ_i and σ_i, as well as on the Lagrangian's parameters ω and λ.

The four series having the general term given in (57), one for each class of paths, do not converge, but are in general Cesaro summable (C, 1). Indeed, a theorem by Weyl [15] shows that the terms of the series

$$\sum_n e^{2\pi i\Phi(n)} \tag{58}$$

are uniformly distributed on the unit circle of the complex plane, if at least one of the coefficients in the polynomial $\Phi(n)$ is irrational. This implies that the partial sums s_n of the series are $O(1)$, i.e. $o(n)$, so that they fluctuate around their (finite) mean value, which is also the limit of the Cesaro means

$$S_n = \frac{s_1 + \ldots + s_n}{n}. \tag{59}$$

The Cesaro summability comes from the cancellation in phase of the contributions, and may be lacking in default of cancellation, which can happen only if all the coefficients of the polynomial $\Phi(n)$ are rational. The Van Vleck series for the semiclassical propagator is summable as well: indeed, each term of the latter, as given by (49), can be written as the sum of the corresponding term of one of the series given in (57) plus a remainder R_{n,δ_i}, which has the same phase as given by (50), but amplitude $O(1/n)$, as easily seen, so that the remainders' series are summable, and even convergent, due to a Hardy's tauberian theorem for Cesaro summability [16]. A detailed comparison between the semiclassical propagator, as computed by resumming its Van Vleck series, and its perturbative series will be given elsewhere [17].

Finally, let us discuss the resonances, i.e. the divergence of one of the terms in the series (1). As we said in section 2, this happens when the final point x_B is conjugate to the initial point x_A with respect to one of the paths, say $x_i(t)$. In this case, a whole family of classical paths, infinitely near to $x_i(t)$, leave x_A and again meet in x_B after time T. All these paths constructively interfere, this giving the divergence of the contribution from the path x_i to the total propagator. Formally, the divergence is due to the second derivative of the action S being infinite at a focal point, which in turn signals a breakdown of the approximation underlying (1). According to the discussion in section 2, when $x_A = 0$, only for the second or the fourth class the final point x_B can be conjugate to the initial one. Moreover, it is easy to see that, given x_A and x_B, there exist two countable sequences of times T_n such that the two points are conjugate with respect to one of the corresponding paths. In fact, the term t_E^B as a function of the energy starts from $-\infty$ a $E = E_{\min}$ and monotonically goes to 0 for $E \to \infty$; in this way, the equation $\delta x = 0$, as given by the appropriate modification of (38), has a solution E_n for each value of the number n of complete oscillations, and the corresponding T_n is then obtained by inserting E_n and n into equations (17) or (19). The first few values of E_n and T_n obtained in this way, with the choice $x_A = 0$, $x_B = 1$, $\omega = 1$, $\lambda = 0.1$, are reported in table 2. From the above discussion about the focal points it follows that the energy values E_n form two sequences converging to $E_{\min}(x_B)$, and when n increases,

Table 2. The energy values E_n and the corresponding times T_n are reported for the first few values of the number n of complete oscillation, for the trajectories such that the final point x_B is conjugate to the initial one x_A. The parameter values are $x_A = 0$, $x_B = 1$, $\omega = 1$, $\lambda = 0.1$.

	Second class		Fourth class	
n	E_n	T_n	E_n	T_n
0	2.309	2.284	1.335	5.129
1	1.024	8.051	0.874	11.007
2	0.785	13.983	0.727	16.971
3	0.686	19.967	0.656	22.971
4	0.634	25.978	0.615	28.990
5	0.603	32.004	0.593	35.021
6	0.584	38.039	0.576	41.059
7	0.575	44.080	0.565	47.102
8	0.561	50.125	0.557	53.149
9	0.554	56.173	0.552	59.198
10	0.549	62.224	0.547	65.249

the times T_n approach $(n+\frac{1}{4})P(E_{\min})$ and $(n+\frac{3}{4})P(E_{\min})$, for the second and the fourth class, respectively. When the initial and the final points are conjugate to each other with respect to a path $x_n(t)$, the contribution of that path to the total propagator dominates the others', and the related probability becomes very large, while for the other paths and previous results, corresponding to generic behaviour, still hold.

In conclusion, there are two possible outcomes for an experience in which one records in a point x_B the energy and the time-of-flight of semiclassical particles, ejected with uniformly distributed energies from a source in x_A and subjected to the anharmonic potential of equation (3): in the generic case the particles arriving after time T will have practically with the same probability, apart from fluctuations corresponding to low values of n, all the energy solutions of equations (16)–(19); otherwise, if time T is equal to one of the times T_n for which the two points are conjugate with respect to a path $x_n(t)$, particles will arrive in x_B preferentially with the energy E_n of that path. Analogous conclusions obviously hold for any potential such that there are many extremals of the action, connecting given points in the same time. An experimental verification of this prediction would give a strong confirmation of the validity of equation (1).

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